Effect of Axial Diffusion of Vorticity on Flow Development in Circular Conduits: Part II. Analytical Solution for Low Reynolds Numbers

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An analytical solution based on a previously proposed entrance model is developed for the case of flow at low Reynolds numbers. Agreement between this solution and the published numerical solution at this Reynolds number limit supports the authenticity of the numerical solutions presented previously.

In the first paper of this series (3), numerical solutions of the complete equations of motion describing flow development of a Newtonian, incompressible fluid in the entrance region were presented for five different Reynolds numbers. By application of an entrance model consisting of an infinite, frictionless stream tube upstream from the conduit entrance, it is possible to extend the analysis of entrance flow development to flow regimes which are inadequately described by a boundary-layer analysis. In this second paper, an analytical solution of the equations describing flow development at low Reynolds numbers in this entrance configuration is presented. Primarily, this work offers an example of an analytical solution for low Reynolds number flow in a closed region. In addition, the good agreement of the numerical and analytical solutions at this limit provides at least partial substantiation of the validity of the numerical solutions presented previously.

DEVELOPMENT OF EQUATIONS

The slow flow form of the vorticity transport equation for this flow field can be deduced from the previously derived vorticity equation (3) by taking the limit as $N_{Re} \to 0$:

$$\frac{\partial^2 \omega}{\partial r^2} + \frac{1}{r}\frac{\partial \omega}{\partial r} - \frac{\omega}{r^2} + \frac{\partial^2 \omega}{\partial z^2} = 0 \tag{1}$$

This equation can also be derived by utilizing a similar limit process on properly constructed dimensionless forms of the Navier-Stokes equations (2) before introducing the vorticity. Furthermore, as before (3) it can be shown that variations in the stream function are described by the equation

$$\frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} = \omega r \tag{2}$$

where the stream function has been defined such that it satisfies the dimensionless continuity equation identically:

$$U = -\frac{1}{r}\frac{\partial \psi}{\partial r} \tag{3}$$

$$V = \frac{1}{r}\frac{\partial \psi}{\partial z} \tag{4}$$

To facilitate the solution of Equations (1) and (2) for vorticity and stream function, the variable ξ is introduced

$$\xi = \omega r \tag{5}$$

and Equations (1) and (2) can thus be converted to the following simpler forms:

$$\frac{\partial^2 \xi}{\partial r^2} - \frac{1}{r}\frac{\partial \xi}{\partial r} + \frac{\partial^2 \xi}{\partial z^2} = 0 \tag{6}$$

$$\frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} = \xi \tag{7}$$

Finally, the boundary conditions for Equations (6) and (7) which describe the infinite entrance flow configuration under consideration can be written as follows:

$$\psi = \frac{\partial \psi}{\partial r} = 0, \ \frac{\partial^2 \psi}{\partial r^2} = \xi \text{ for } r = 1, \ z > 0 \tag{8}$$

$$\psi = \frac{1}{2}, \ \xi = 0 \text{ for } r = 0, \ -\infty < z < \infty \tag{9}$$

$$\psi = \xi = 0 \text{ for } r = 1, \ z < 0 \tag{10}$$

$$\psi = \frac{1}{2}(1 - r^2), \ \xi = 0 \text{ for } z = -\infty, \ 0 \leqq r \leqq 1 \tag{11}$$

$$\psi = r^2\left(\frac{r^2}{2} - 1\right) + \frac{1}{2}, \ \xi = 4r^2 \text{ for } z = \infty, \ 0 \leqq r \leqq 1 \tag{12}$$

SOLUTION OF EQUATIONS FOR $z < 0$

Because of the nature of the boundary conditions for this particular flow field, it is necessary to obtain separate solutions to Equations (6) and (7) for the vorticity and stream function distributions for the regions $z < 0$ and $z > 0$. The solutions for these two regions are then joined at $z = 0$ by the imposition of continuity conditions for the stream function and vorticity. For $z < 0$, introduction of the variable $\hat{\psi}$, defined by the equation

$$\psi = \frac{1}{2}(1 - r^2) + \hat{\psi} \tag{13}$$

gives

$$\frac{\partial^2 \hat{\psi}}{\partial z^2} + \frac{\partial^2 \hat{\psi}}{\partial r^2} - \frac{1}{r}\frac{\partial \hat{\psi}}{\partial r} = \xi \tag{14}$$

for Equation (7) and

$$\hat{\psi} = \xi = 0 \text{ for } r = 1, \ z < 0 \tag{15}$$

$$\hat{\psi} = \xi = 0 \text{ for } r = 0,\ z < 0 \tag{16}$$

$$\hat{\psi} = \xi = 0 \text{ for } z = -\infty,\ 0 \leqq r \leqq 1 \tag{17}$$

for the boundary conditions.

The solution to Equations (6) and (14), two elliptic partial differential equations, is most easily obtained by solving first the homogeneous equation for ξ, substituting this result in Equation (14), and then solving the nonhomogeneous equation for $\hat{\psi}$. We proceed by assuming a solution to Equation (6) of the form

$$\xi(r, z) = R(r)\, Z(z) \tag{18}$$

and applying the method of separation of variables. Substitution of Equation (18) into Equation (6) gives the following two ordinary differential equations:

$$\frac{d^2R}{dr^2} - \frac{1}{r}\frac{dR}{dr} + \alpha^2 R = 0 \tag{19}$$

$$\frac{d^2Z}{dz^2} - \alpha^2 Z = 0 \tag{20}$$

Solution of these two equations and introduction of the boundary conditions yield

$$\xi = \sum_{n=1}^{\infty} E_n\, r\, J_1(\alpha_n r) \exp(\alpha_n z) \tag{21}$$

where the eigenvalues α_n are the positive roots of

$$J_1(\alpha_n) = 0 \tag{22}$$

The solution has been written as a Fourier-Bessel series so that an infinite set of constants E_n is available for the matching of the solutions at $z = 0$.

To solve Equation (14) with Equation (21) substituted for the right side of the equation, we assume a solution for the n^{th} eigenfunction of the following form:

$$\hat{\psi}_n(r, z) = r\, J_1(\alpha_n r)\, \hat{Z}_n(z) \tag{23}$$

Substitution of Equation (23) into Equation (14) gives the differential equation

$$\frac{d^2\hat{Z}_n}{dz^2} - \alpha_n^2\, \hat{Z}_n = E_n \exp(\alpha_n z) \tag{24}$$

whose solution subject to the boundary conditions can be combined with Equation (23) to give the following result:

$$\hat{\psi} = \sum_{n=1}^{\infty} F_n\, r\, J_1(\alpha_n r) \exp(\alpha_n z) + \sum_{n=1}^{\infty} \frac{E_n}{2\alpha_n} r\, J_1(\alpha_n r)\, z \exp(\alpha_n z) \tag{25}$$

Again, the solution has been written as a Fourier-Bessel series to provide another infinite set of constants F_n for satisfying the continuity conditions at $z = 0$.

Consequently, it is evident from Equations (3), (5), (13), (21), and (25) that the vorticity, stream function, and axial velocity distributions for $z < 0$ are given by the following equations:

$$\omega = \sum_{n=1}^{\infty} E_n\, J_1(\alpha_n r) \exp(\alpha_n z) \tag{26}$$

$$\psi = \frac{1}{2}(1 - r^2) + \sum_{n=1}^{\infty} F_n\, r\, J_1(\alpha_n r) \exp(\alpha_n z) + \sum_{n=1}^{\infty} \frac{E_n}{2\alpha_n} r\, J_1(\alpha_n r)\, z \exp(\alpha_n z) \tag{27}$$

$$U = 1 - \sum_{n=1}^{\infty} F_n\, J_o(\alpha_n r)\, \alpha_n \exp(\alpha_n z) - \sum_{n=1}^{\infty} \frac{E_n}{2} J_o(\alpha_n r)\, z \exp(\alpha_n z) \tag{28}$$

SOLUTION OF EQUATIONS FOR $z > 0$

The mathematical problem becomes more complex in the downstream region of the flow field, because a Cauchy condition for the stream function is imposed at the real tube wall in addition to the problem of matching solutions at $z = 0$. Consequently, to overcome these difficulties it is necessary to split the problem in the downstream region into two distinct problems and then combine the results to give the desired solution. Thus, for the region $z > 0$, it is convenient to let

$$\psi = \frac{1}{2} + r^2\left(\frac{r^2}{2} - 1\right) + \psi^* + \bar{\psi} \tag{29}$$

$$\xi = 4r^2 + \xi^* + \bar{\xi} \tag{30}$$

Substitution of Equations (29) and (30) into Equations (6), (7), (8), (9), and (12) yields the following two sets of differential equations and boundary conditions:

$$\frac{\partial^2\psi^*}{\partial z^2} + \frac{\partial^2\psi^*}{\partial r^2} - \frac{1}{r}\frac{\partial\psi^*}{\partial r} = \xi^* \tag{31}$$

$$\frac{\partial^2\xi^*}{\partial r^2} - \frac{1}{r}\frac{\partial\xi^*}{\partial r} + \frac{\partial^2\xi^*}{\partial z^2} = 0 \tag{32}$$

$$\psi^* = \xi^* = 0,\ \frac{\partial\psi^*}{\partial r} = g(z) \text{ for } r = 1,\ z > 0 \tag{33}$$

$$\psi^* = \xi^* = 0 \text{ for } r = 0,\ z > 0 \tag{34}$$

$$\psi^* = \xi^* = 0 \text{ for } z = \infty,\ 0 \leqq r \leqq 1 \tag{35}$$

$$\frac{\partial^2\bar{\psi}}{\partial z^2} + \frac{\partial^2\bar{\psi}}{\partial r^2} - \frac{1}{r}\frac{\partial\bar{\psi}}{\partial r} = \bar{\xi} \tag{36}$$

$$\frac{\partial^2\bar{\xi}}{\partial r^2} - \frac{1}{r}\frac{\partial\bar{\xi}}{\partial r} + \frac{\partial^2\bar{\xi}}{\partial z^2} = 0 \tag{37}$$

$$\bar{\psi} = 0,\ \frac{\partial\bar{\psi}}{\partial r} = -g(z),\ \frac{\partial^2\bar{\psi}}{\partial r^2} = -g(z) + \bar{\xi} \quad \text{for } r = 1,\ z > 0 \tag{38}$$

$$\bar{\psi} = \bar{\xi} = 0 \text{ for } r = 0,\ z > 0 \tag{39}$$

$$\bar{\psi} = \bar{\xi} = 0 \text{ for } z = \infty,\ 0 \leqq r \leqq 1 \tag{40}$$

The function $g(z)$ is to be determined during the course of the analysis.

The solution of Equations (31) and (32) subject to Equations (33), (34), and (35) can be obtained by a procedure completely analogous to that used in deriving the solution to Equations (6) and (14) subject to Equations (15), (16), and (17). The results of such an analysis are simply

$$\xi^* = \sum_{n=1}^{\infty} M_n\, r\, J_1(\alpha_n r) \exp(-\alpha_n z) \tag{41}$$

$$\psi^* = \sum_{n=1}^{\infty} N_n\, r\, J_1(\alpha_n r) \exp(-\alpha_n z)$$

$$- \sum_{n=1}^{\infty} \frac{M_n}{2\alpha_n} r J_1(\alpha_n r) z \exp(-\alpha_n z) \quad (42)$$

where the Fourier-Bessel series in these equations contain two more infinite sets of constants M_n and N_n, which are to be used in matching the solutions for $z < 0$ and $z > 0$. The eigenvalues α_n are again defined by Equation (22). Also, it is evident from Equations (33) and (42) that $g(z)$ is defined by the following equation:

$$g(z) = \left(\frac{\partial \psi^*}{\partial r}\right)_{r=1} = \sum_{n=1}^{\infty} N_n \alpha_n J_o(\alpha_n) \exp(-\alpha_n z)$$

$$- \sum_{n=1}^{\infty} \frac{M_n}{2} J_o(\alpha_n) z \exp(-\alpha_n z) \quad (43)$$

Furthermore, substitution of an assumed solution of the form

$$\bar{\xi}(r, z) = \bar{R}(r) \bar{Z}(z) \quad (44)$$

into Equation (37) leads to the following two ordinary differential equations:

$$\frac{d^2\bar{R}}{dr^2} - \frac{1}{r}\frac{d\bar{R}}{dr} - \beta^2 \bar{R} = 0 \quad (45)$$

$$\frac{d^2\bar{Z}}{dz^2} + \beta^2 \bar{Z} = 0 \quad (46)$$

Introduction of the solutions to Equations (45) and (46) into Equation (44) and utilization of the boundary conditions for $\bar{\xi}$ give

$$\bar{\xi} = \int_0^{\infty} r I_1(\beta r) [T(\beta) \cos \beta z + Q(\beta) \sin \beta z] \, d\beta \quad (47)$$

where $T(\beta)$ and $Q(\beta)$ must be determined from the boundary conditions at the real tube wall. The integral form of Equation (47) represents the inclusion in the expression for $\bar{\xi}$ of an infinite number of solutions to Equation (37) corresponding to all possible positive values of β, nonintegral as well as integral. To obtain the solution to Equation (36) with Equation (47) substituted for the right side of the equation, we assume the following form for an eigenfunction of the solution which corresponds to one of the eigenvalues in the continuous spectrum of eigenvalues:

$$\bar{\psi}(r, z) = [T(\beta) \cos \beta z + Q(\beta) \sin \beta z] \bar{R}(r) \quad (48)$$

Substitution of Equation (48) into Equation (36) gives the differential equation

$$\frac{d^2\bar{R}}{dr^2} - \frac{1}{r}\frac{d\bar{R}}{dr} - \beta^2 \bar{R} = r I_1(\beta r) \quad (49)$$

whose solution can be expressed as

$$\bar{R} = C_1 r I_1(\beta r) + C_2 r K_1(\beta r) + \frac{r^2 I_2(\beta r)}{2\beta} + \frac{r I_1(\beta r)}{2\beta^2} \quad (50)$$

Introduction of this result into Equation (48) and evaluation of C_1 and C_2 from the boundary conditions give the following solution for $\bar{\psi}$ in integral form:

$$\bar{\psi} = \int_0^{\infty} \left[\frac{r^2 I_2(\beta r) I_1(\beta) - r I_1(\beta r) I_2(\beta)}{2\beta I_1(\beta)}\right] \times [T(\beta) \cos \beta z + Q(\beta) \sin \beta z] \, d\beta \quad (51)$$

Equations (47) and (51) satisfy all of the boundary conditions except the equations for the first and second derivatives of the stream function at the real tube wall. If it is possible to choose $T(\beta)$ and $Q(\beta)$ so that $\partial\bar{\psi}/\partial r = -g(z)$ at $r = 1$, then both of these boundary conditions will clearly be satisfied since they are really not independent restrictions. This point has been discussed previously (3). From Equation (51) it follows that

$$-g(z) = \left(\frac{\partial \bar{\psi}}{\partial r}\right)_{r=1} = \int_0^{\infty} \left[\frac{I_1^2(\beta) - I_o(\beta) I_2(\beta)}{2 I_1(\beta)}\right] \times [T(\beta) \cos \beta z + Q(\beta) \sin \beta z] \, d\beta \quad (52)$$

Now we consider a Fourier integral representation of $-g(z)$ which is defined for $z \geqq 0$ and is to be continued as an even function for negative values of z. For these restrictions it is known (1) that the Fourier integral for this function reduces to

$$-g(z) = \frac{2}{\pi} \int_0^{\infty} \cos \beta z \, d\beta \int_0^{\infty} [-g(q)] \cos \beta q \, dq \quad (53)$$

Comparison of Equations (52) and (53) shows that

$$Q(\beta) = 0 \quad (54)$$

$$T(\beta) = \frac{4 I_1(\beta) \int_0^{\infty} [-g(q)] \cos \beta q \, dq}{\pi[I_1^2(\beta) - I_o(\beta) I_2(\beta)]} \quad (55)$$

In addition, from Equation (43), which defines $g(z)$ explicitly, it can be shown that

$$\int_0^{\infty} [-g(q)] \cos \beta q \, dq = - \sum_{n=1}^{\infty} \frac{N_n \alpha_n^2 J_o(\alpha_n)}{\alpha_n^2 + \beta^2} + \sum_{n=1}^{\infty} \frac{M_n (\alpha_n^2 - \beta^2) J_o(\alpha_n)}{2(\alpha_n^2 + \beta^2)^2} \quad (56)$$

so that the Fourier integral representation of $-g(z)$ actually takes the form

$$- \sum_{n=1}^{\infty} N_n \alpha_n J_o(\alpha_n) \exp(-\alpha_n z) + \sum_{n=1}^{\infty} \frac{M_n}{2} J_o(\alpha_n) z \exp(-\alpha_n z)$$

$$= - \sum_{n=1}^{\infty} \frac{2N_n \alpha_n^2 J_o(\alpha_n)}{\pi} \int_0^{\infty} \frac{\cos \beta z}{\alpha_n^2 + \beta^2} d\beta + \sum_{n=1}^{\infty} \frac{M_n J_o(\alpha_n)}{\pi} \int_0^{\infty} \frac{\cos \beta z (\alpha_n^2 - \beta^2)}{(\alpha_n^2 + \beta^2)^2} d\beta \quad (57)$$

From Equations (3), (5), (29), (30), (41), (42), (47), (51), (54), (55), and (56) it follows that the vorticity, stream function, and axial velocity distributions for $z > 0$ are given by the following equations:

$$\omega = 4r + \sum_{n=1}^{\infty} M_n J_1(\alpha_n r) \exp(-\alpha_n z) - \sum_{n=1}^{\infty} \frac{4N_n \alpha_n^2 J_o(\alpha_n)}{\pi} \int_0^{\infty} \frac{\cos \beta z \, I_1(\beta r) \, I_1(\beta)}{(\alpha_n^2 + \beta^2) P(\beta)} d\beta + \sum_{n=1}^{\infty} \frac{2M_n J_o(\alpha_n)}{\pi} \int_0^{\infty} \frac{(\alpha_n^2 - \beta^2) \cos \beta z \, I_1(\beta r) \, I_1(\beta)}{(\alpha_n^2 + \beta^2)^2 P(\beta)} d\beta \quad (58)$$

$$\psi = \frac{1}{2} + r^2\left(\frac{r^2}{2} - 1\right) + \sum_{n=1}^{\infty} N_n\, r\, J_1(\alpha_n r)\exp(-\alpha_n z)$$

$$- \sum_{n=1}^{\infty} \frac{M_n}{2\alpha_n} r\, J_1(\alpha_n r)\, z \exp(-\alpha_n z)$$

$$- \sum_{n=1}^{\infty} \frac{2N_n \alpha_n^2 J_o(\alpha_n)}{\pi} \int_0^{\infty} \frac{\cos\beta z\, W(r,\beta)}{(\alpha_n^2 + \beta^2)\,\beta\, P(\beta)}\, d\beta$$

$$+ \sum_{n=1}^{\infty} \frac{M_n J_o(\alpha_n)}{\pi} \int_0^{\infty} \frac{\cos\beta z\,(\alpha_n^2 - \beta^2) W(r,\beta)}{(\alpha_n^2 + \beta^2)^2\,\beta\, P(\beta)}\, d\beta \quad (59)$$

$$U = 2(1 - r^2) - \sum_{n=1}^{\infty} N_n J_o(\alpha_n r)\,\alpha_n \exp(-\alpha_n z)$$

$$+ \sum_{n=1}^{\infty} \frac{M_n}{2} J_o(\alpha_n r)\, z \exp(-\alpha_n z)$$

$$+ \sum_{n=1}^{\infty} \frac{2N_n \alpha_n^2 J_o(\alpha_n)}{\pi} \int_0^{\infty} \frac{\cos\beta z\, S(r,\beta)}{(\alpha_n^2 + \beta^2)\, P(\beta)}\, d\beta$$

$$- \sum_{n=1}^{\infty} \frac{M_n J_o(\alpha_n)}{\pi} \int_0^{\infty} \frac{\cos\beta z(\alpha_n^2 - \beta^2) S(r,\beta)}{(\alpha_n^2 + \beta^2)^2\, P(\beta)}\, d\beta \quad (60)$$

$$P(\beta) = I_1^2(\beta) - I_o(\beta)\, I_2(\beta) \quad (61)$$

$$W(r,\beta) = r^2 I_2(\beta r)\, I_1(\beta) - r\, I_1(\beta r)\, I_2(\beta) \quad (62)$$

$$S(r,\beta) = r\, I_1(\beta r)\, I_1(\beta) - I_o(\beta r)\, I_2(\beta) \quad (63)$$

MATCHING OF SOLUTIONS AT $z = 0$

The solution to the flow problem is thus complete except for the evaluation of the four sets of constants, E_n, F_n, M_n, and N_n, from the continuity conditions for the vorticity and the stream function at $z = 0$. By requiring that these variables and their first derivatives be continuous at $z = 0$, we impose the following conditions:

$$\omega(r, 0^-) = \omega(r, 0^+) \quad (64)$$

$$\left(\frac{\partial\omega}{\partial z}\right)_{z=0^-} = \left(\frac{\partial\omega}{\partial z}\right)_{z=0^+} \quad (65)$$

$$\psi(r, 0^-) = \psi(r, 0^+) \quad (66)$$

$$\left(\frac{\partial\psi}{\partial z}\right)_{z=0^-} = \left(\frac{\partial\psi}{\partial z}\right)_{z=0^+} \quad (67)$$

From Equations (26), (58), and (65) it is evident that

$$E_n = -M_n \quad (68)$$

TABLE 1. CALCULATED COEFFICIENTS

n	M_n	N_n
1	—4.7510	0.26210
2	4.4923	—0.071725
3	—4.3779	0.033579
4	4.2836	—0.019457
5	—4.1685	0.012718
6	4.0234	—0.0089206
7	—3.8576	0.0065771
8	3.6978	—0.0049782
9	—3.5478	0.0038442
10	3.4112	—0.0030303

TABLE 2. COMPARISON OF ANALYTICAL AND FINITE-DIFFERENCE SOLUTIONS

Axial velocities at $z = -0.13378$

r	U, finite-difference solution for $N_{Re} = 0$	U, analytical solution
0	1.5796	1.5821
0.1	1.5648	1.5708
0.2	1.5303	1.5368
0.3	1.4735	1.4800
0.4	1.3937	1.4001
0.5	1.2900	1.2961
0.6	1.1613	1.1668
0.7	1.0060	1.0102
0.8	0.82385	0.82391
0.9	0.62744	0.61881
1.0	0.47651	0.50197

and from Equations (27), (59), (67), and (68) it can be shown that

$$F_n = -N_n \quad (69)$$

Furthermore, substitution of Equations (26) and (58) into Equation (64), multiplication by $r\, J_1(\alpha_m r)\, dr$, and integration between the limits 0 and 1 yield

$$M_m + \frac{4}{\alpha_m J_2(\alpha_m)}$$

$$- \sum_{n=1}^{\infty} \frac{4N_n \alpha_n^2 J_o(\alpha_n)\,\alpha_m}{\pi J_2(\alpha_m)} \int_0^{\infty} \frac{I_1^2(\beta)\, d\beta}{(\alpha_n^2 + \beta^2)\,(\alpha_m^2 + \beta^2) P(\beta)}$$

$$+ \sum_{n=1}^{\infty} \frac{2M_n J_o(\alpha_n)\,\alpha_m}{\pi J_2(\alpha_m)} \int_0^{\infty} \frac{(\alpha_n^2 - \beta^2)\, I_1^2(\beta)\, d\beta}{(\alpha_n^2 + \beta^2)^2 (\alpha_m^2 + \beta^2) P(\beta)} = 0 \quad (70)$$

Similarly, substitution of Equations (27) and (59) into Equation (66), multiplication by $J_1(a_m r) dr$, and integration between the limits 0 and 1 give

$$N_m - \frac{4}{\alpha_m^3 J_2(\alpha_m)}$$

$$+ \sum_{n=1}^{\infty} \frac{4N_n \alpha_n^2 J_o(\alpha_n)\,\alpha_m}{\pi J_2(\alpha_m)} \int_0^{\infty} \frac{I_1^2(\beta)\, d\beta}{(\alpha_n^2 + \beta^2)\,(\alpha_m^2 + \beta^2)^2 P(\beta)}$$

$$- \sum_{n=1}^{\infty} \frac{2M_n J_o(\alpha_n)\,\alpha_m}{\pi J_2(\alpha_m)} \int_0^{\infty} \frac{(\alpha_n^2 - \beta^2)\, I_1^2(\beta)\, d\beta}{(\alpha_n^2 + \beta^2)^2 (\alpha_m^2 + \beta^2)^2 P(\beta)} = 0 \quad (71)$$

Equations (70) and (71) actually represent two infinite systems of linear equations for the two sets of infinitely many unknowns, M_n and N_n. An iterative successive approximation method of solution can be used to obtain the coefficients from finite forms of the two infinite sets of simultaneous equations. The first ten coefficients of each set are presented in Table 1 as examples of the calculation.

Values of vorticity, stream function, and axial velocity obtained from this analytical solution are in good agreement with the values generated by the numerical solution. A typical example of this agreement is illustrated in Table 2 where a comparison is given between the axial velocities calculated from the analytical solution and those derived from the finite-difference solution. The axial velocity distribution obtained from the analytical solution is thus represented to within a few percent by the finite-difference distribution presented in Figure 3 of reference 3. This figure shows that the analytical solution is applicable when the axial diffusion of vorticity overwhelms the convective transport and significant changes in the axial velocity occur before the fluid enters the conduit. The good agreement between these two solutions at this Reynolds number limit is considered to be evidence in support of the accuracy of the numerical solutions at other Reynolds numbers since the same numerical scheme was employed in all cases.

NOTATION

$g(z)$ = function as defined by Equation (43)
$I_n(\lambda)$ = modified Bessel function of first kind of order n
$J_n(\lambda)$ = Bessel function of first kind of order n
$K_n(\lambda)$ = modified Bessel function of second kind of order n
N_{Re} = Reynolds number
r = radial distance/radius of tube
U = axial component of velocity/average velocity in tube
V = radial component of velocity/average velocity in tube
z = axial distance/radius of tube

Greek Letters

ξ = variable as defined by Equation (5)
ψ = dimensionless stream function
ω = dimensionless component of vorticity vector

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The Control of Nonlinear Systems. Part I:

Direct Search on the Performance Index

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A straightforward and simple algorithm is presented for obtaining the optimal control of nonlinear systems. This algorithm has many of the advantages of dynamic programming but not the disadvantage of excessive computer storage. A number of numerical examples are presented to show the versatility of the method.

In the recent chemical engineering literature there has been an overabundance of papers on the development of computer algorithms suitable for control of nonlinear systems. Each of these papers has used some form of dynamic programming (*2, 10*) or the maximum principle (*4, 5, 7, 12, 15*). Of the two methods the former would seem to appeal more to engineers, since dynamic programming has the very important advantage that trajectory control and state constraints may be included in a simple fashion.

Unfortunately dynamic programming tends to flounder on the curse of dimensionality in which computer storage requirement rises exponentially with the number of state variables. Thus a three-state variable problem may require more high-speed storage than is currently available in the largest digital computers. At the same time it should be pointed out that considerable care must be used in both dynamic programming and the maximum principle or the results of an optimal control calculation may be in error (*8*).

In this paper we wish to develop a simple search technique for solving the optimal control problem, which retains many of the excellent features of dynamic programming but which eliminates the storage disadvantage. The primary idea of the method is to trade off computer storage for computing time, an almost necessary requirement for developing practical methods of solution of nonlinear problems.

The explicit method exploited here fits into the category of solution defined by Bellman as an approximation in policy space (*2*) and by Dreyfus as an approximation to the solution (*6*). Rather than requiring the use of derivatives to improve the control function (*5, 12*), the method is an application of finding a minimum along one or more coordinate directions at a time by using a direct search. Converse (*3, 4*) was the first to use this method in control problems but his procedure was a specific and special form of the algorithm used here. As a result his work has not received the attention that it warrants.

To illustrate the basic details of the algorithm the famous cross-current extraction system with recycle, a sequence of nonisothermal CSTR's and a plate type of gas absorber system will be analyzed and computational results will be presented.

PROBLEM DEFINITION

We consider in this paper systems which are described by the set of nonlinear differential equations and associated boundary conditions:

$$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t)] \tag{1}$$

$$\mathbf{x}(0) \quad \text{given} \tag{2}$$

Alternately we may conceive of a discretized form of Equation (1)

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